DETERMINATION OF LOCAL STRESSES IN A CYLINDRICAL SHELL LOADED OVER A CIRCULAR AREA

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A cylindrical shell is related to structural elements which are used extensively. Therefore, calculation of its local strength has received considerable attention. The most studied stressed state is with shell loading over a rectangular area and section of the coordinate line [1-3]. There is much less information for strength analysis of a shell in the zone of a circular area of loading. The first closed equations for calculating bending moments under normal force in the center of a circle were obtained in [4]. Later these values were represented by a series in MacDonald complex argument functions [5-7] which made it possible to expand the range of their application into shells of other shapes. In order to compute forces and moments in panels of zero and positive curvature power series with a logarithm were suggested in [8]. By cutting off these expansions simple asymptotic equations were given in which shell size and the area of loading were considered, and a rule was also given for external load distribution. Distribution density is prescribed by a power relationship with arbitrary indices. By altering this it is possible to obtain both regular and singular distributions. However, the application limits for the asymptotic expressions obtained remained unknown.

Closed equations are suggested in this work for calculating bending moments and tangential forces, which is important in complete determination of normal stresses at the outer and inner surfaces of the shell. Calculation of them is reduced to tabulated Thomson functions. Apart from uniform distribution consideration was given to a parabolic load distribution with a zero pressure value at the contour of the area. Simple asymptotic equations are given for calculating forces and moments, and limits of their applicability were established. A comparison with the numerical results of other authors is given. It is shown that the local stressed state of infinitely long shells is determined by one dimensionless similarity parameter. This makes it possible to simplify universal curves for designing shells of different thickness and diameters loaded over circular areas of different radii. Stress determination at the most critical point (center of the area) with a prescribed distribution density for load is reduced to using four of these curves.

In analyzing the local stressed state we start from equations for four thin elastic isotropic shells with a large variability index [9]. The area of loading is assumed to be quite distant from the ends of a thin-walled body when it is possible to ignore their effect on the value of local stresses. Under these conditions the study is carried out conveniently by the method of two-dimensional Fourier transforms. The solution obtained will not be periodic over the circular coordinate. However, this does not give large errors in view of the rapid decrease in solutions with respect to this variable [10]. In favor of these approximate solutions is the conclusion made in comparing them with accurate solutions in [11].

In order to calculate tangential forces t_1 and t_2 and bending moments m_1 and m_2 at the center of a circle with uniformly distributed force P by the method of integral Fourier transforms quadratures have been obtained [8]

$$m_1 = A_1 + \nu A_2,\ m_2 = A_2 + \nu A_1; \tag{1}$$

$$t_1\delta_{1j} + t_2\delta_{2j} + A_1\delta_{1k} + A_2\delta_{2k} = -\frac{qr}{2\pi}\int\int_{-\infty}^{\infty} [b^4 R_2 \xi^2 (\eta^2\delta_{1j} + \xi^2\delta_{2j}) -$$

$$- (\xi^2+\eta^2)^2 (\xi^2\delta_{1k} + \eta^2\delta_{2k}) J_0\left(r\sqrt{\xi^2+\eta^2}\right)] [(\xi^2+\eta^2)^4 + b^4\xi^4]^{-1}\, d\xi d\eta,$$
$$b^4 = 12(1-\nu^2) h^{-2} R_2^{-2}. \tag{2}$$

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here R_2 and h are shell radius and thickness; ν is Poisson's ratio of its material; $q = P(2\pi r)^{-1}$ is distribution density for external force P over a circle of radius r; $J_0(z)$ is first-order Bessel function of a zero series; δ_{mj} and δ_{nk} are Kronecker symbols; $m = \overline{1, 2}$; $n = \overline{1, 2}$.

We substitute in (2) variables ξ and η for γ and φ assuming $\xi = \gamma \cos\varphi$, $\eta = \gamma \sin\varphi$. In new variables the proper integral with respect to γ is expressed in terms of a MacDonald function $K_0(z)$ [12]:

$$\int_0^\infty \frac{J_0(\gamma r)\gamma}{\gamma^2 + ib^2\cos^2\varphi} d\gamma = K_0(\sqrt{i}\, br\cos\varphi), \quad i = \sqrt{-1}.$$

Integration with respect to φ is also reduced to tabulated integrals [12]

$$\int_0^{\pi/2} K_0(z\cos\varphi)\, d\varphi = \frac{\pi}{2} I_0(z) K_0(z), \quad \int_0^{\pi/2} K_0(z\cos\varphi)\cos 2\varphi d\varphi = -\frac{\pi}{2} I_1(z) K_1(z),$$

in which I_n and K_n are modified Bessel functions of the power n.

As a result of this forces and moments at the center of the circle are represented by the expressions

$$\begin{aligned} t_{1,2} &= \frac{\sqrt{3(1-\nu^2)}}{h} qr \operatorname{Im}[I_0(ar)K_0(ar) \pm I_1(ar)K_1(ar)], \\ A_{1,2} &= \frac{qr}{2} \operatorname{Re}[I_0(ar)K_0(ar) \mp I_1(ar)K_1(ar)], \quad a = \frac{1}{2} b\sqrt{i}. \end{aligned} \tag{3}$$

These are functions of the effect for determing t_j and m_j at the most critical point, i.e., the center of a circular area of loading by a distributed normal force. With uniform distribution of force over a circle of radius R we have to integrate them from zero to R, which it is easy to carry out by means of the expressions

$$\int_0^R rI_0(ar)K_0(ar)\, dr = \frac{R^2}{2}[I_0(aR)K_0(aR) + I_1(aR)K_1(aR)],$$

$$\int_0^R rI_1(ar)K_1(ar)\, dr = \frac{R^2}{2}\left[I_1(aR)K_1(aR) + I_0(aR)K_2(aR) - \frac{2}{a^2R^2}\right].$$

Then the real and imaginary parts should be separated by using the relationships [12] $I_n(x\sqrt{i}) = e^{-n\pi i/2}[\operatorname{ber}_n(x) + i\operatorname{bei}_n(x)]$, $K_n(x\sqrt{i}) = e^{n\pi i/2}[\operatorname{ker}_n(x) + i\operatorname{kei}_n(x)]$ between modified Bessel functions of the complex argument and Thomson functions. As a result of this forces and moments at the center of the circle with uniform pressure $q = P(\pi R^2)^{-1}$ are presented in closed form

$$\begin{aligned} t_{1,2} = \frac{P\sqrt{3(1-\nu^2)}}{2\pi h} &[B_0T_0 + C_0K_0 + B_1T_1 + \\ &+ C_1K_1 \pm (B_1T_1 + C_1K_1 - B_0T_2 - C_0K_2 + 8b^{-2}R^{-2})], \\ A_{1,2} = \frac{P}{4\pi}&[B_0K_0 - C_0T_0 + B_1K_1 - C_1T_1 \mp (B_1K_1 - C_1T_1 + C_0T_2 - B_0K_2)]. \end{aligned} \tag{4}$$

Here and below use is made of abbreviated notations for Thomson functions $B_m = \operatorname{ber}_m(\rho)$, $C_m = \operatorname{bei}_m(\rho)$, $K_m = \operatorname{ker}_m(\rho)$, $T_m = \operatorname{kei}_m(\rho)$, $\rho = bR/2$.

We note that expressions for bending moments in the form of (1) and (4) are suggested in [4] by integrating fundamental solutions for a concentrated force. We obtain them by a simpler method. From a physical point of view distributions are more real with a zero value of pressure at the contour of the area. Therefore, we stop at a parabolic distribution $f(r) = q(1 - r^2R^{-2})$, $q = 2P(\pi R^2)^{-1}$.

In order to calculate t_j and m_j expression (3) should now be integrated from zero to R with load f(r). This does not cause any difficulties since

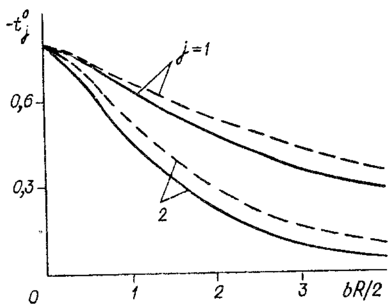

Fig. 1

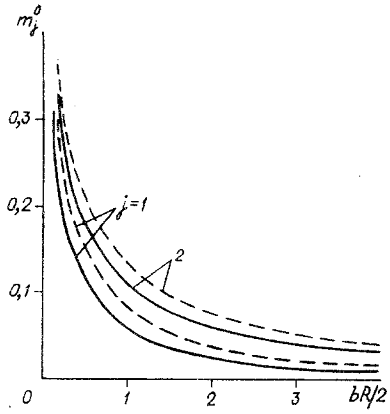

Fig. 2

TABLE 1

x	f_1 From curves in [6, 7]	f_2 From curves in [6, 7]	f_1 From Eqs. (1) and (5)	f_2 From Eqs. (1) and (5)	x	f_1 From curves in [6, 7]	f_2 From curves in [6, 7]	f_1 From Eqs. (1) and (5)	f_2 From Eqs. (1) and (5)
1	1,72	2,05	1,70	2,04	6	0,63	0,93	0,64	0,95
2	1,30	1,60	1,28	1,61	8	0,50	0,78	0,49	0,79
4	0,86	1,18	0,86	1,19	10	0,38	0,67	0,40	0,68

$$\int_0^R r^3 I_0(ar)\,K_0(ar)\,dr = \frac{R^4}{6}\Big[I_0(aR)\,K_0(aR) + \frac{1}{aR} I_1(aR)\,K_0(aR) + I_1(aR)\,K_1(aR) - \frac{1}{aR} I_2(aR)\,K_1(aR)\Big],$$

$$\int_0^R r^3 I_1(ar)\,K_1(ar)\,dr = \frac{R^4}{6}[I_1(aR)\,K_1(aR) + I_2(aR)\,K_2(aR)].$$

After calculating the quadratures and separating the imaginary and real parts we find

$$t_{1,2} = \frac{2P\sqrt{3(1-\nu^2)}}{3\pi h}\Big\{B_0T_0 + C_0K_0 + B_1T_1 + C_1K_1 + \frac{1}{\sqrt{2}\,bR}(B_0T_1 + C_0K_1 + B_0K_1 - C_0T_1) - \frac{1}{\sqrt{2}\,bR}(B_1K_2 - C_1T_2 + C_1K_2 + B_1T_2) \pm$$
$$\pm\Big[B_1T_1 + C_1K_1 - B_0T_2 - C_0K_2 + \frac{\sqrt{2}}{bR}(B_1K_2 - C_1T_2 + B_1T_2 + C_1K_2) + 12(bR)^{-2}\Big]\Big\},$$

$$A_{1,2} = \frac{P}{3\pi}\Big\{B_0K_0 - C_0T_0 + B_1K_1 - C_1T_1 + \frac{1}{\sqrt{2}\,bR}(B_0K_1 - C_0T_1 - C_0K_1 - B_0T_1) - \frac{1}{\sqrt{2}\,bR}(B_1K_2 - C_1T_2 - C_1K_2 - B_1T_2) \mp$$
$$\mp\Big[B_1K_1 - C_1T_1 - B_0K_2 + C_0T_2 + \frac{\sqrt{2}}{bR}(B_1K_2 - C_1T_2 - B_1T_2 - C_1K_2)\Big]\Big\}.$$

We introduce dimensionless forces $t_j^0 = 2\pi h\left(P\sqrt{3(1-\nu^2)}\right)^{-1} t_j$ and moments $m_j^0 = P^{-1} m_j$ ($j = \overline{1, 2}$). From the solutions constructed it follows that with prescribed ν values of m_j^0 and t_j^0 are only determined by one dimensionless parameter $\rho = bR/2$, i.e., the similarity parameter, since shells with different R_2, h, R with fixed ρ have identical t_j^0 and m_j^0. Therefore, in the case of long cylindrical shells when it is possible to ignore the effect of ends on the distribution of local stresses, graphical dependences of t_j^0 and m_j^0 on ρ take a universal character and they may be used for the design of shells with different R_2, h, and R. These dependences are presented in Figs. 1 and 2 (solid lines relate to uniform load distribution, and broken lines to parabolic distribution with $\nu = 0.3$).

It can be seen that for fixed values of ρ a parabolic distribution is more critical, i.e., it leads to greater stresses at the center of the circle. Tangential force t_1 appears to be greater than t_2, which was also noted with numerical summing of a trigonometrical series for a square loading area [2]. With an increase in ρ the ratio of moments m_1/m_2 tends toward ν. With ρ = 4 it differs insignificantly from 0.3. This ratio is adopted in semi-momentless shell theory [2], and therefore it is used for calculations with ρ > 4.

With small values of ρ calculation of t_j^0 and m_j^0 may be carried out by means of elementary equations without drawing on tables [13] of Thomson functions. These equations are obtained by cutting off power series in the solutions in [8]. With uniform distribution of the external force the first two terms of the expansion give

$$t_{1,2}^0 = -\left\{\frac{\pi}{4} + \frac{1}{8}\rho^2\left[(2 \mp 1)\ln\frac{\rho}{2} - (0.3456 \pm 0.0772)\right]\right\},$$
$$P^{-1}A_{1,2} = -\frac{1}{8\pi}\left[2\ln\frac{\rho}{2} + 0.1544 \pm 1 - \frac{\pi(2 \pm 1)}{16}\rho^2\right]. \tag{5}$$

In the case of a parabolic distribution in the two-term approximation we find

$$t_{1,2}^0 = -\left\{\frac{\pi}{4} + \frac{1}{12}\rho^2\left[(2 \mp 1)\ln\frac{\rho}{2} - (0.6790 \mp 0.0895)\right]\right\},$$
$$P^{-1}A_{1,2} = -\frac{1}{8\pi}\left[2\ln\frac{\rho}{2} - 0.3456 \pm 1 - \frac{\pi(2 \pm 1)}{24}\rho^2\right].$$

Comparison of the results for calculating m_j^0 and t_j^0 by asymptotic equations shows that with bR ≤ 1.8 the error of elementary equations does not exceed the error for technical shell theory (of the order of hR_2^{-1} compared with unity). Therefore inequality bR < 1.8 should be considered as the region for permissible use of asymptotic equations without forgetting that calculation by shell theory is only possible in principle with R ≥ 0.68h [5, 6].

In conclusion we compare the results of calculations by (1) and (5) with calculations in a computer [6, 7] with ν = 0.3, $R_2h^{-1} = 100$, R = xh. Shown in the second and third columns of Table 1 are values of $f_j = 6m_j$ borrowed for different x from curves in [6, 7], and in the fourth and fifth columns there are values of f_j calculated by Eqs. (1) and (5). Deviations of numbers obtained by the two methods are small and are due to a certain extent to the error in graphical information. Consequently, the asymptotic equations proposed provide acceptable accuracy and they are convenient for engineering calculations in the range of change in similarity parameter indicated. It is noted that approximate closed solutions may also be obtained by the synthesis method [14] for the shell stressed state.

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COMBINATION OF RAYLEIGH AND DYNAMIC EDGE EFFECT METHODS IN STUDYING VIBRATIONS OF RECTANGULAR PLATES

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The dynamic edge effect method (DEEM) suggested by Bolotin has been used extensively in solving problems of natural vibrations for elastic rectangular plates and also for structures consisting of them [1]. Generally speaking the method intended for finding high natural frequencies and forms under kinetic boundary conditions also gives good results for low forms of vibration [2]. With existence of static conditions at the contour the accuracy of determining low natural values decreases [3]. The error of the DEEM is connected with the fact that the solution constructed by means of it does not satisfy the original problem in the vicinity of boundaries. One possible way for refining the method is construction of angular boundary layers [4], and another is combination of the asymptotic method with variation methods. Combination of the DEEM with the Rayleigh–Ritz method was the subject of [5], although there only the case of kinematic boundary conditions was considered, and therefore it is difficult to check the efficiency of this approach. Equations obtained in [5] for natural frequencies are only applicable for a square plate clamped along all edges. Of particular interest with this combination is estimation of the first approximation (Rayleigh equation) since in this case it is possible to obtain an expression for natural frequency in closed form.

In the present work an asymptotic expression is obtained by combining Rayleigh and DEEM methods for the frequency of natural vibrations suitable for arbitrary unchanged conditions at the boundary along the rectilinear edge, and the efficiency of this approach has been studied.

We consider vibration of an elastic rectangular ($0 \le x_1 \le a_1$, $0 \le x_2 \le a_2$) plate. According to Rayleigh the expression for frequency parameter λ has the form

$$\lambda = a_1 a_2 \left[\left(\rho h/D\right) \int\limits_0^{a_1} \int\limits_0^{a_2} \left(w_{,11}^2 + w_{,22}^2 + 2\nu w_{,11} w_{,22} + 2(1-\nu) w_{,12}^2\right) dx_1 dx_2 \Big/ \left(\int\limits_0^{a_1} \int\limits_0^{a_2} w^2 dx_1 dx_2 \right) \right]^{1/2}. \tag{1}$$

Here $\lambda = \omega a_1 a_2 (\rho h/D)^{1/2}$; w is normal deflection; ν is Poisson's ratio; ω is natural frequency; h is thickness; ρ is material density; $D = Eh^3/[12(1-\nu^2)]$; E is Young's modulus.

The expression for the function of deflection obtained by means of the DEEM [1] is written as

$$f(x_1, x_2) = S_1(x_1) \sin(\beta_1 x_2 + l_2) + S_2(x_2) \sin(\beta_1 x_1 + l_1); \tag{2}$$

$$S_i(x_i) = \sin(\beta_i x_i + l_i) + C_{i1} \exp(\alpha_i x_i) + C_{12} \exp(-\alpha_i x_i) \ (i = 1, 2). \tag{3}$$

We take the expression for deflection $w(x_1, x_2)$ in the form

$$w(x_1, x_2) = S_1(x_1) S_2(x_2). \tag{4}$$

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